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# Representatives for $P$-Typical Curves*

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0. A new construction of the $p$-hyperexponential (Witt) bialgebras is given, along with generalizations, indexed by families of rational primes. Previous constructions have proceeded by putting an exotic comultiplication on a polynomial algebra [2, 7, 12] with nontrivial divisibility problems and/or rather mysterious, though elegant, transformations of coordinates. In the construction described below, roughly speaking, the coalgebra structure is always visible and the polynomial algebra structure, with adjunct divisibility problems solved. emerges as a rather easy consequence of the fundamental theorem of symmetric functions. Ditters has announced a construction of these bialgebras [4], and noncommutative analogs. He uses a generalization of the Artin–Hasse exponential series, familiar in the $p$-hyperexponential case. His proofs will appear shortly [5].

We have capitalized on the traditional view of algebraic topologists that it is worthwhile thinking of the Hopf algebra $H^*(BU$ as an algebra of symmetric functions; these bialgebras are described as quotients by ideals generated by certain symmetric powers. $H^*(BU)$ itself is given a novel presentation. Although no applications are presented here, we have used these results to describe the action of the Steenrod algebra on $H^*(BU; Z_p)$ [16], by formulas which appear centrally in Lazard's axiomatic treatment of commutative formal groups [8, 9]. Lazard suppresses, in a calculated manner, a role that these bialgebras might have played. Up to typification, at least, this role is played below. At this point, a generalization of Husemoller's [6] decomposition of $H^*(BU)$ into a tensor product of hyperexponential bialgebras is proved in a new and simple fashion.

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1. If $P$ is a set of primes in $N$, the natural numbers, and $Z_p$ is the integers localized at $P$, let the polynomial algebra $Z_p[c] \equiv Z_p[c_{i,j}]_{(i,j)\in N\times N}$ be given the unique coalgebra structure extending

$$\Delta c_{i,j} = \sum_{j'+j''=j} c_{i,j'} \otimes c_{i,j''} + c_{i,j} \otimes 1 + 1 \otimes c_{i,j} \qquad (i,j) \in N \times N. \quad (1.1)$$

Let $J_P$ be the Hopf) ideal in $Z_P[c]$ generated by

$$\begin{aligned} &(R_{i,j}) \quad jc_{i,j} - \sum_{j'+j''=j} c_{i,j'}c_{ij'',1} - c_{ij,1} \qquad (i,j) \in N \times N, \\ &(S_i) \quad c_{i,1}, \qquad i \notin P^*, \text{ the submonoid of } N \text{ generated by } P. \end{aligned} \quad (1.2)$$

$Z_P[c]/J_P$ will have torsion, in general. The bialgebra $B_P(Z_P)^m \equiv$ (by definition) $((Z_P[c]/J_P)/\text{torsion})$ can be graded so that $c^p_{i,j} \in B_P(Z_P)^p_{2ijm}$, if $c^p_{i,j}$ is the image of $c_{i,j}$ under the canonical surjection, and consequently [10] has an antipode, $\psi$, of order 2. If $R$ is a $Z_P$ algebra, define $B_P(R)^m \equiv B_P(Z_P)^m \otimes R$. If $n \in P^*$, let ${}_nB_P(R)^m$ be the sub-Hopf algebra generated by $\{c^p_{i,j}\}_{ij\leqslant n}$.

THEOREM 1. *$B_P(R)^m$ is a polynomial algebra over $R$, freely generated by $\{c^p_{1,j}\}_{j\in P^*}$. ${}_nB_P(R)^m$ is the subalgebra generated by $\{c^p_{1,j}\}_{j\in P^*, j\leqslant n}$.*

*Proof.* It suffices to consider the case $R = Z_p$. An easy induction shows that $c^p_{i,j} = 0$ if $i \notin P^*$. Suppose $i \in P^*$. $jc^p_{i,j} - \sum_{j'+j''=j} c^p_{i,j'}c^p_{ij'',1} = ijc^p_{1,ij} - \sum_{k+l=ij} c^p_{1,k}c^p_{l,1}$, so if $j \mid \sum_{j'+j''=j} c^p_{i,j'}c^p_{ij'',1} - \sum_{k+l=ij} c^p_{1,k}c^p_{l,1}$ whenever $j \in N$, an easy induction shows that $\{c^p_{1,k}\}_{k\in N}$ is a generating set for $B_P(Z_P)^m$.

Suppose $n = n_{P'}n_P$, $n_{P'} \in P'^*$ and $n_P \in P^*$, $n_{P'} \neq 1$.

$$n_{P'}n_Pc^P_{1,n} - \sum_{k+l=n} c^P_{1,k}c^P_{l,1} = n_Pc^P_{n_{P'},n_P} - \sum_{k+l=n_P} c^P_{n_{P'},k}c^P_{n_{P'}l,1},$$

so if $n_P \mid \sum_{k+l=n_p} c^p_{n_p,k}c^p_{n_p,l,1} - \sum_{k+l=n_p n_p} c^p_{1,k}c^p_{l,1}$ (since $n_{P'}$ is a unit in $Z_P$) $\{c^p_{1,k}\}_{k\in P^*}$ is a generating set for $B_P(Z_P)^m$. Moreover $c^p_{i,j} = c^p_{i,j}(c^p_{1,k})_{k\leqslant ij}$ if $i \neq 1$ or $j \notin P^*$.

It is clear from the algebra presentation that $\{c^p_{i,1}\}_{i\in P^*}$ is a set of polynomial generators for $B_P(Q)^m$, and it follows immediately that $\{c^p_{k,1}\}_{k\in P^*}$ freely generates if it generates $B_P(Z_P)^m$.

The following lemma completes the argument.

LEMMA 1. *If $(i,j) \in N \otimes N$, $\sum_{j'+j''=j} c^p_{i,j'}c^p_{ij'',1} - \sum_{k+l=ij} c^p_{1,k}c^p_{l,1}$ is divisible by $j$.*

*Proof.* Let $n \in N$ satisfy $ij \leqslant n$. Let $\alpha_{ij} \in Z[X_1{}^i, X_2{}^i,..., X_n{}^i] \equiv Z[X^i] \subseteq Z[X]$ be the signed elementary symmetric function $(-1)^{(i-1)j}\sigma_j(X^i)$. If $k = lm$, $\sigma_{k,1} \equiv (-1)^{\nu-1}S_k(X) = (-1)^{lm-1}S_m(X^l)$, a signed symmetric power.

Comparing Newton's formulas in $Z[X]$ and $Z[X^i] \subseteq Z[X]$,

$$\begin{aligned} j\sigma_{i,j} - \sum_{j'+j''=j} \sigma_{i,j'}\sigma_{ij'',1} &= (-1)^{(i-1)j} j\sigma_j(X^i) \\ &\quad - \sum_{j'+j''=j} (-1)^{(i-1)j'} (-1)^{ij''-1} \sigma_{j'}(X^i)\, S_{j''}(X^i) \\ &= (-1)^{(i-1)j} (-1)^{j-1} S_j(X^i) = (-1)^{ij-1} S_{ij}(X) \\ &= ij\sigma_{ij}(X) - \sum_{k+l=ij} (-1)^{l-1} \sigma_k(X)\, S_l(X) \\ &= ij\sigma_{1,ij} - \sum_{k+l=m} \sigma_{1,k}\sigma_{l,1}\,. \end{aligned} \tag{1.3}$$

Consequently, $\sigma_{i,j} = i\sigma_{1,ij} + 1/j\{\sum_{j'+j''=j} \sigma_{i,j'}\sigma_{ij'',1} - \sum_{k+l=ij} \sigma_{1,k}\sigma_{l,1}\}$ in $Z[\sigma_{1,m}]_{m\leqslant n} \subseteq Z[X]$. The lemma is a formal consequence of the relations obtained from the Newton's formulas by replacing $\sigma_{r,s}$ by $c^p_{r,s}$ which hold in $B_P(R)^m$, by inspection of the generators of $J_P$. It must be noted that the Newton formulas provide an algorithm for expressing $\sigma_{i,j}$ as a polynomial in $\{\sigma_{1,k}\}_{k\leqslant n}$. Since the expression must be the only one, the divisibility must follow from divisibility of coefficients.

THEOREM 2. *If $\iota_k$ is the inclusion of the subcoalgebra $\{c^p_{k,j}\}_{j\in N}$ in $B_P(R)^m$ $\bigotimes_{k\in P^*} \iota_k$ is an isomorphism of coalgebras. Equivalently, $\{\prod_{i\in P^*} c^p_{i,j'} \mid j \in \prod_{i\in P^*} Z^+\}$ is a basis for $B_P(R)^m$.*

*Proof.* Suppose the elements of the proposed basis of degree $< n$ span a pure submodule. We may assume $R = Z_P$. Let $p \in P$, $p \mid \sum_{\sum j_i = n} a_j \prod_{i\in P^*} c^p_{i,j_i}$, and $p \mid a_j \to a_j = 0$. From the inductive assumption and an application of the diagonal map, it appears that $p \mid a_j$ unless $j_i = \delta_i^n$ and $n \in P^*$. If $p \nmid c_{n,1}$, the linear combination is zero.

LEMMA 2. *Let $V_d$: $B_P(R)^m \to B_P(R)^{md}$ be defined by $V_d(c^p_{1,j}) = c^p_{1,j/d}$ if $d \mid j$ and $j \in P^*$, and 0 if $d \nmid j$ and $j \in P^*$. $V_d$ is a Hopf algebra map and $V_d(c^p_{j,1}) = dc^p_{j/d,1}$ if $d \mid j$ and 0 otherwise.*

*Proof.* Since $\{c^p_{i,j}\}_{j\in P^*}$ is a set of polynomial generators, $V_d$ is indeed defined. Assume $R = Z_P$, that $V_d(c^p_{1,j}) = c^p_{1,j/d}$ (or 0) whether or not $j \in P^*$, if $j < k$, and that $V_d(c^p_{j,1}) = dc^p_{j/d,1}$ (or 0) if $j < k$. If $k \in P^*$,

$$\begin{aligned} &V_d \left(kc^P_{1,k} - \sum_{k'+k''=k} c^P_{1,k'}c^P_{k'',1}\right) \\ &\qquad = d \left(k/dc^P_{1,k/d} - \sum_{k'+k''=k} c^P_{1,k'/d}c^P_{k''/d,1}\right) = dc^P_{k/d,1}\,. \end{aligned}$$

If $k \notin P^*$, $V_d(0) = 0 = V_d(c^p_{k,1}) = kV_d(c^p_{1,k}) - d\sum_{k'+k''} c^p_{1,k'/d}c^p_{k''/d,1}$. Consequently, $k/dV_d(c^p_{1,k}) - \sum_{l'+l''=l} c^p_{1,l'} c^p_{l'',1} = c^p_{k/d,1}$, and $V_d(c^p_{1,k}) = c^p_{1,k/d}$. Since $V_d$ restricts to the subcoalgebra $\{1\} \cup \{c^p_{1,k}\}_{k\in N}$ as a coalgebra map, and the polynomial generators are contained in this subcoalgebra, $V_d$ is a Hopf algebra map.

Returning to the proof of Theorem 2, if $n \in P^*$, and $n$ is not a power of $p \in P$, let $q \in P$, $q \mid n$. $p \mid c^p_{n,1} \Rightarrow p \mid V_f(c^p_{n,1}) = qc^p_{n/q,1}$. This is impossible by the inductive assumption. On the other hand, if $n$ is a power of $p$, we take advantage of a classical formula in symmetric functions:

$$c^P_{k,1} = \sum_{\substack{mij_i=k \\ mj_i=j}} (-1)^{(j-1)} (k/j) \binom{j}{j_1, j_2, \ldots, j_k} (c^P_{1,1})^{j_1} (c^P_{1,2})^{j_a} \cdots (c^P_{1,k})^{j_k}. \quad (1.4)$$

*Proof.* By induction on $k$, using only relations of type (i) in $J_P$. (This is Waring's formula, expressing symmetric powers as polynomials in elementary symmetric functions, by the analogy pressed in Lemma 1.) The application is: $p \mid (p^r/\text{g.c.d.}(j_i)) \cdot (\text{g.c.d.}(j_i)/j)\binom{j}{j_1 \ldots j_{p^r}}$ if $\sum_1^{p^r} ij_i = p^r$ and $\sum_1^{p^r} j_i = j$ unless $j_i = \delta_1{}^i \cdot p^r$. If $p \mid c^p_{p^r,1}$, then $p \mid (c^p_{1,1})\, p^r$. But $c^p_{1,1}$ is a polynomial generator of $B_P(Z_P)^m$, and this cannot happen.

It follows that the proposed basis spans a pure submodule of $B_P(Z_P)^m$. Since the one-to-one correspondence $\prod_{i\in P^*} (c^p_{1,i})^{j_i} \leftrightarrow \prod_{i \in P^*} c^p_{i,j_i}$ is degree preserving and $B_P(Z_P)^m$ has finite type, this submodule cannot be proper.

2. Let $H^R_{pf}$ be the category of commutative, cocommutative $R$ Hopf algebras whose coalgebras are filtered inductive limits of subcoalgebras which are finitely generated projective, and whose inclusions have $R$-linear retractions. Let $B_P(R)$ be any of the Hopf algebras $B_P(R)^m$, rid of its filtration, and given instead the primitive filtration; i.e., $c^p_{i,j} \in B_P(R)_j - B_P(R)_{j-1}$. Morris and Pareigis [11] have provided a theory of formal groups over discrete rings, $R$, in which the categories $H^R_{pf}$ play the role of flat commutative formal groups.

If $R$ is a $Z_p$ algebra, a $P$-typical curve in $G \in H^R_{pf}$ is an element of $H^R_{pf}(B_P(R), G)$. With operations convolution and composition, and convolution identity 0, defined by $0(c^p_{1,i}) = 0$, $i \in P^*$, the $P$-typical curves form a ring. Let $\text{Cart}_P(R)$ denote the opposite ring structure on the $P$-typical curves. By composition and convolution, the $P$-typical curves in $G$ are a module over $\text{Cart}_P(R)$.

3. We shall parametrize the rings $\text{Cart}_P(R)$, introducing the rings of Witt vectors in a natural way, as in the treatments of Dieudonne [2], and Ditters [3].

If $m \in P^*$, let $V_m \in \text{Cart}_P(R)$ be defined as in Lemma 2: $V_m(c^p_{1,j}) \equiv c^p_{1,j/m}(0)$ if $m \mid j$ $(m \nmid j)$ and $j \in P^*$. If $n \in P^*$, let $F_n \in \text{Cart}_P(R)$ be defined by $F_n(c_{1,j}) = c_{n,j}$, $j \in P^*$. If $r \in R$, let $[r] \in \text{Cart}_P(R)$ be defined by $[r](c^p_{1,j}) = r^j c^p_{1,j}$ if $j \in P^*$. Cursory examination of the generators of $J_P$ reveals the identities:

$$\begin{aligned}
&\text{(i)} \quad V_m(c^P_{1,j}) = c^P_{1,j/m}(0) && \text{if} \quad m \mid j \ (m \nmid j);\\
&\text{(ii)} \quad V_m(c^P_{i,1}) = mc^P_{i/m,1}(0) && \text{if} \quad m \mid i \ (m \nmid i);\\
&\text{(iii)} \quad F_n(c^P_{i,j}) = c^P_{in,j} && \text{if} \quad (i,j) \in P^* \times N;\\
&\text{(iv)} \quad [r](c^P_{1,j}) = r^j c^P_{1,j} && \text{if} \quad j \in N.
\end{aligned} \tag{3.1}$$

The following relations are easily verified in $\mathrm{Cart}_P(R)$ (whose multiplicative structure is opposite to composition).

$$\begin{aligned}
&\text{(i)} \quad V_1 = F_1 = [1_R] = 1;\\
&\text{(ii)} \quad V_m V_n = V_{mn}\,, F_m F_n = F_{mn}\,, [a][b] = [ab] \text{ if } m, n \in P^*,\ a, b \in R;\\
&\text{(iii)} \quad \text{if } m, n \in P^* \text{ and g.c.d.}(m, n) = 1,\ V_m F_n = F_n V_m;\\
&\text{(iv)} \quad \text{for all } n \in P^*, F_n V_n = n \cdot 1;\\
&\text{(v)} \quad [c]\, V_n = V_n[c^n] \text{ and } F_n[c] = [c^n]F_n \text{ for all } n \in P \text{ and } c \in R.
\end{aligned} \tag{3.2}$$

That is, (iv) is proved as follows: If $G$ is any connected Hopf algebra, $n \cdot 1_G$, restricted to the primitive submodule of $G$, is multiplication by $n$. In $B_P(Z_P)$, $V_n(c^p_{nj,1}) = nc^p_{j,1} = n \cdot 1(c^p_{j,1})$. Assume $V_n(c_{n,k}) = n \cdot 1(c_{1,k})$ if $k < j$. $n \cdot 1(c^p_{j,1}) = V_n(c^p_{nj,1}) = jV_n(c^p_{n,j}) - n \cdot 1(\sum_{j'+j''=j} c^p_{1,j'}c^p_{j'',1})$. So $jV_n(c^p_{n,j}) - jn1(c^p_{1,j}) = 0$. There is no torsion in $B_P(Z_P)$, so $V_n(c^p_{n,j}) = n \cdot 1(c^p_{1,j})$. The corresponding result for arbitrary $Z_P$ algebras, $R$, follows by naturality. (iv) is an immediate consequence.

If $G \in H^K_{pf}$, $H^R_{pf}(B_P(R), G)$ is topologized as follows: If $U_i = \{f \mid f(c^p_{1,j}) = 0$ if $j \in P^*, j < i\}$ whenever $i \in P^*$, $\{U_i\}_{i \in P^*}$ is a neighborhood base of 0. $\mathrm{Cart}_P(R)$ is thereby given the structure of a topological ring, which is easily shown to be complete.

Let $\mathrm{ord}_{P,R}(G)\colon H^R_{pf}(B_P(R), G) \to P^* \cup \{\infty\}$ be defined as follows: $\mathrm{ord}(f) = 0$ if $f \in U_i - U_j$ if $j > i$. That is, $\{\{f \mid \mathrm{ord}(f) \geqslant i\}\}_{i \in P^*}$ is a neighborhood base of 0.

The following relations are satisfied in $\mathrm{Cart}_P(R)$.

$$\begin{aligned}
&\text{(i)} \quad \mathrm{ord}(x \pm y) > \min(\mathrm{ord}(x), \mathrm{ord}(y)),\\
&\text{(ii)} \quad \mathrm{ord}(xy) \geqslant \mathrm{ord}(x),\\
&\text{(iii)} \quad \mathrm{ord}([a] + [b] - [a + b]) > 1.
\end{aligned} \tag{3.3}$$

The proofs follow immediately from the definitions.

THEOREM 3. *Every element of* $\mathrm{Cart}_P(R)$ *can be expressed uniquely as* $\sum_{m,n \in P^*} V_m[d_{m,n}]F_n$. *The* $d_{m,n} \in R$ *satisfies the condition: For each* $m \in P^*$, $d_{m,n} = 0$ *for all but finitely many n. Moreover,* $\mathrm{ord}(\sum_{m,n \in P^*} V_m[d_{m,n}]F_n] = m_{\inf}$, *the least m such that* $d_{m,n} \neq 0$ *for some n.* $m \in P^*$, $d_{m,n} = 0$ *for all but finitely many n.*

*Proof.* Let $f \in \mathrm{Cart}_P(R)$. $f(c^p_{1,1})$ is primitive, and from Theorem 2, must be a linear combination, $\sum_{n \in P^*} d_{1,n}c^p_{n,1}$, $d_{1,n} = 0$ for all but finitely many $n$. Let

$m_1 = 1, m_2, m_3, \ldots$ be the integers in $P^*$ in their natural order $f - \sum_{n\in P^*} F_n \circ [d_{1,n}] \circ V_1$ and 0 agree on $\{c^p_{1,i}\}_{1\leqslant i\leqslant m_2}$, since $c_{1,i}$ is a polynomial in $c^p_{1,1}$ if $1 \leqslant i < m_2$. Consequently, $(f - \sum_{n\in P^*} F_n \circ [d_{1,n}] \circ V_1)(c^p_{1,m_2})$ is primitive, a linear combination $\sum_{n\in P^*} d_{m_2,n} c^p_{n,1} = (\sum_{n\in P^*} F_n \circ [d_{m_2,n}] \circ V_{m_2})(c^p_{1,m_2})$, and $d_{m,n_2} = 0$ for all but finitely many $n \in P^*$. $f - \sum F_n \circ [d_{1,n}] \circ V_1 - \sum F_n \circ [d_{m_2,n}] \circ V_{m_2}$ and 0 agree on $\{c^p_{1,i}\}_{1\leqslant i\leqslant m_3}$, since $c^p_{1,i}$ is a polynomial in $\{c^p_{1,1}, c^p_{1,m_2}\}$ if $1 \leqslant i < m_3$. Continuing in this manner, we find $f - \sum_{j\in N} \sum_{n\in P^*} F_n \circ [d_{m_j,n}] \circ V_{m_j} = 0$. The formula for ord is clear.

Lazard [9] proves that properties (3.2)(i)–(v), (3.3)(i)–(iii), and Theorem 3 characterize $\mathrm{Cart}_P$ as a functor from commutative rings to complete topological rings. We shall no prove this here; our concern has been to show that these characteristic properties have elementary Hopf algebraic interpretations. In [8], property (3.3)(ii) was improperly omitted. (This weakness was pointed out to us by the referee.)

It is clear from the proof of Theorem 3 that the endomorphisms of $B_P(R)^m$ can be identified with the subring of $\mathrm{Cart}_P(R)$ of elements of the sort $\sum_{n\in P^*} V_n[d_n]\, V_n$. With this identification, if $j \in P^*$,

$$\Big(\sum_{n\in P^*} V_n[d_n]\, F_n\Big)(c_{1,j}) = \Big(\sum_{n|j} n d_n^{j/n}\Big)\, c^P_{1,j} + \text{decomposables},$$
$$\Big(\sum_{n\in P^*} V_n[d_n]\, F_n\Big)(c^P_{j,1}) = \Big(\sum_{n|j} n d_j^{j/n}\Big)\, c^P_{j,1}. \tag{3.4}$$

Thus, the endomorphisms of $B_P(R)^m$ can be identified with the $P$-Witt vectors with coefficients in $R$ (see [1]), for convolution and composition, restricted to the primitive submodule, $\coprod_{j\in P^*} (Rc^p_{j,1})$, are additive and multiplicative on components.

An easy consequence of this representation of Witt vectors is the following: A $P$-Witt vector is invertible if and only if its phantom components are invertible.

More generally, if $j \in P^*$,

$$\Big(\sum_{m,n\in P^*} V_m[d_{m,n}]\, F_n\Big)(c^P_{i,j}) = \sum_{i\in P^*}\Big(\sum_{l|(i,j)} i/l\, d^l_{j/l,i/l}\Big)(c^P_{1,i}) + \text{decomposables},$$
$$\Big(\sum_{m,n\in P^*} V_m[d_{m,n}]\, F_n\Big)(c^P_{j,1}) = \sum_{i\in P^*}\Big(\sum_{l|(i,j)} j/l\, d^l_{j/l,i/l}\Big)(c^P_{i,1}). \tag{3.5}$$

An induction on the filtrands of the primitive filtration proves the following: $\sum_{m,n\in P^*} V_m[d_{m,n}]\, F_n \in \mathrm{Cart}_P(R)$ is invertible if and only if the matrix $(\sum_{l|(i,j)} j/l\, d^l_{j/l,i/l})_{(i,j)\in P^*+P^*}$ is invertible. Moreover, the inverse of such a matrix is a matrix of the same type.

4. If $P_1 \Sigma P_2$ and $R$ is a $Z_{P_2}$ algebra, it is also a $Z_{P_1}$ algebra. Let $J_{P_1,P_2}$ be the ideal in $B_{P_1}(Z_{P_2})$ generated by $\{c^{p_1}_{n,1}\}_{n\in P_1{}^*-P_2{}^*}$. It is clear from the definitions of $J_{P_1}$ and $J_{P_2}$ that $(B_{P_1}(Z_{P_2})/J_{P_1,P_2})/\text{torsion} \otimes_{Z_{p_2}} R \simeq B_{P_2}(Z_{P_2}) \otimes_{Z_{p_2}} R = B_{P_2}(R)$.

A more interesting relationship between $B_{P_2}(R)$ and $B_{P_1}(R)$ exists. The following Theorem 5 is due to Husemoller [6], in case $P_1 =$ all primes, $P_2 = \{p\}$, by a very different proof. Generalizations, in the spacial cases, are due to Newman [13], and Ravenel and Wilson [14] who show that bipolynomial Hopf algebras over a $Z_{\{p\}}$ algebra $R$ are tensor products of $B_{\{p\}}(R)$'s. Newman treats noncommutative cases analogously. It is surprising that no such theorem is possible for nonempty $P$'s other than singletons (see [15], which gives a best approximation).

If $i \in (P_1 - P_2)^*$, let $\nu_i \colon B_{P_2}(R) \to B_{P_1}(R)$ be the algebra map defined by $\nu_i(c^p_{1,j}) = F_i \circ \prod_{p \in P_1 - P_2} (1 - p^{-1} \circ F_p \circ V_p)(c^p_{1,j})$, if $j \in P_2{}^*$.

THEOREM 4. *$\nu_i$ is a map of Hopf algebras.*

*Proof.* $(1 - p^{-1} \circ F_p \circ V_p)(c^p_{j,1}) = c^p_{j,1} - p^{-1} p c^{p_1}_{j/p,p \cdot 1} = 0$ if $p \mid j$ and $c^{p_1}_{j,1}$ if $p \nmid j$. It follows that $F_i \circ \prod_{p \in P_1 - P_2} (1 - p^{-1} \circ F_p \circ V_p)$ induces a Hopf algebra map from $(B_{P_1}(Z_{P_2})/J_{P_1,P_2})/\text{torsion} \otimes_{Z_{p_2}} R$ into $B_{P_1}(R)$. This must be $\nu_i$, since the maps agree on generators.

THEOREM 5. *$\bigotimes_{i \in (P_1 - P_2)^*} \nu_i \colon \bigotimes_{i \in (P_1 - P_2)^*} B_{P_2}(R)^{mi} \to B_{P_1}(R)^m$ is an isomorphism of Hopf algebras if $R$ is a $Z_{P_2}$ algebra and $P_1 \supseteq P_2$.*

*Proof.* $\nu_i(c^{p_2}_{1,j}) = i c^{p_1}_{1,ij} +$ decomposables, if $i \in (P_1 - P_2)^*$ and $j \in P_2{}^*$. This follows from (i) $\prod_{p \in P_1 - P_2} (1 - p^{-1} \circ F_p \circ V_p)$ is an endomorphism of $B_{P_1}(R)^m$; (ii) as shown in Section 3, if $f$ is such an endomorphism, $f(c_{1,k}) = \lambda_{f,k} c_{1,k} +$ decomposables, and $f(c_{k,1}) = \bar{g}_{f,k} c_{k,1}$, then $\lambda_{f,k} = \bar{g}_{f,k}$; (iii) from the proof of Theorem 4, if $f$ is $\prod_{p \in P_1 - P_2} (1 - p^{-1} \circ F_p \circ V_p)$, $\lambda_{f,k} = \bar{g}_{f,k} = 1$ if $k \in P_2{}^*$, and 0 if $k \notin P_2{}^*$; (iv) $F_i(c_{1,k}) = c_{i,k} = i c_{1,ik} +$ decomposables.

Since $\{c^{p_2}_{i,1,j}\}_{i \in (P_1 - P_2)^*, j \in P_2{}^*}$ is a complete set of polynomial generators for $\bigotimes_{i \in (P_1 - P_2)^*} B_{P_2}(R)^{mi}$, and $\{i c_{1,ij}\}_{i \in (P_1 - P_2)^*, j \in P_2{}^*}$ is a complete set of polynomial generators for $B_{P_1}(R)^m$, $\bigotimes_{i \in (P_1 - P_2)^*} \nu_i$ is an isomorphism.

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